COMPLETE VECTOR FIELDS ON $(\mathbb{C}^*)^n$

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Abstract. We prove necessary and sufficient conditions for a rational vector field on $(\mathbb{C}^*)^n$ to be complete.

1. Introduction

This paper has been motivated by the desire to understand the group of holomorphic automorphisms of certain complex manifolds M. In a number of cases the group, $\mathrm{Aut}(M)$, is known to be a finite dimensional Lie group: this is notably so when M is a bounded domain in $\mathbb{C}^n$ or, more generally, a hyperbolic manifold; or when M is compact, see [Kob70]. The manifolds that we are concerned with are not such. Little is known about the automorphism groups of nonhyperbolic affine varieties beyond the fact that they can be huge ([Var]); but for the case $M = \mathbb{C}^n$ see [And90, AL92, RR88, For96].

If $2n$ generic planes are removed from $\mathbb{C}^n$ the resulting space is known to be hyperbolic (see [Blo26, Gre77]), and therefore the automorphism groups are finite dimensional. The case of $\mathbb{C}^n$ minus m hyperplanes ($n < m < 2n$) is subject to research by the author. In this paper we take M to be $\mathbb{C}^n$ minus n hyperplanes in general position ($n \geq 2$). If $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we can take M to be $(\mathbb{C}^*)^n$. These manifolds are known to have infinite dimensional automorphism groups. In the case $n = 2$ we can get automorphisms of $(\mathbb{C}^*)^2$ by taking an arbitrary holomorphic function f, two integers n_1, n_2 and complex numbers c_1, c_2 and forming

$$(z_1, z_2) \mapsto (z_1 e^{n_2 f(z_1^{n_1} z_2^{n_2})}, z_2 e^{-n_1 f(z_1^{n_1} z_2^{n_2})}). \tag{1}$$

Once we observe that this mapping preserves $z_1^{n_1} z_2^{n_2}$ we easily see that it is bijective. This verifies the claim that the automorphism groups of $(\mathbb{C}^*)^n$ are infinite dimensional. Other automorphisms of $(\mathbb{C}^*)^2$ are given by

$$(z_1, z_2) \mapsto (z_1^{a_{11}} z_2^{a_{12}}, z_1^{a_{21}} z_2^{a_{22}}), \tag{2}$$

where the integers a_{ij} satisfy $a_{11}a_{22} - a_{12}a_{21} = 1$. It is conjectured that these mappings generate the full automorphism group of $(\mathbb{C}^*)^2$. Nishimura [Nis92] proves that any automorphism of $(\mathbb{C}^*)^2$ which extends to $\mathbb{C}^2$ and preserves the volume form $dz_1 \wedge dz_2$ is of the form (1), with $n_1 = n_2 = 1$. He also has results about automorphisms of $\mathbb{C} \times \mathbb{C}^*$ — see [Nis86].

Peschl [Pes56] claimed to have proved that all automorphisms of $(\mathbb{C}^*)^2$ that extend to mappings of $\mathbb{C}^2$ preserve the volume form

$$\frac{dz_1 \wedge dz_2}{z_1 z_2},$$

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but his proof has been found to be incomplete. Accordingly, there is a conjecture that any automorphism of $(\mathbb{C}^*)^2$ preserves this form.

An action of $\mathbb{C}$ on M is a family $\{\phi_t : t \in \mathbb{C}\}$ of automorphisms of M such that $\phi_s \circ \phi_t = \phi_{s+t}$. Suzuki [Suz78, Suz77] has studied actions on a two dimensional manifold. He proves among other things that if an action of $\mathbb{C}$ on $(\mathbb{C}^*)^2$ factors through an action of $\mathbb{C}^*$ on $(\mathbb{C}^*)^2$ then the action is by linear mappings $(z_1, z_2) \mapsto (e^{n_2 t} z_1, e^{-n_1 t} z_2)$.

An action is generated by a complete holomorphic vector field. (A holomorphic vector field is said to be complete if all integral curves are entire holomorphic functions, see [For96].) In this paper we give a complete characterization of complete holomorphic rational vector fields on $(\mathbb{C}^*)^n$. Such fields can be written

$$\dot{z}_i = z_i p_i(z)$$

where $z = (z_1, \ldots, z_n)$ and p_i are Laurent polynomials in $z_1, \ldots, z_n$, that is, polynomials in these variables and their inverses. As a corollary, we prove the above conjecture for automorphisms coming from such fields.

2. Preliminaries

We will need the following elementary facts from Nevanlinna theory (see e.g. [Lan87]). For any meromorphic function of one complex variable $f(z)$ we set

$$\begin{gathered} m(r,f) = \frac{1}{2\pi r} \int_{|z|=r} \log^+ |f(z)|\,|dz|, \\ N(r,f) = \sum_{f(a)=\infty,\ a\neq 0} \log^+ \left|\frac{r}{a}\right| + k \log r \quad \text{and} \\ T(r,f) = m(r,f) + N(r,f), \end{gathered}$$

where k is the order of the pole of f at 0. ($k = 0$ if f is regular at 0.) In the definition of N, the sum is taken over all poles a of f, with regard to multiplicity. The function $T(r,f)$ is called the Nevanlinna characteristic of f. The following properties are easily verified.

$$\begin{gathered} T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + O(1) \\ T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2), \\ T(r, f^d) = dT(r, f) \quad (d > 0). \end{gathered} \tag{3}$$

Here and in the sequel, the estimates $O(g)$ and $o(g)$ are as $r \to \infty$. The first fundamental theorem of Nevanlinna theory says that

$$T(r, 1/f) = T(r,f) + O(1).$$

We also have the Lemma of the logarithmic derivative (LLD),

$$m(r, f'/f) = o_{\text{excl}}(T(r,f)).$$

Here o_{excl} means that the estimate holds outside a set of finite measure. We also use the corresponding notation O_{excl}. From LLD and the preceding inequalities it follows that

$$m(r, f^{(k)}) \leq (1 + o_{\text{excl}}(1)))T(r,f) \tag{4}$$

$$m(r, f^{(k)}/f) = o_{\text{excl}}(T(r,f)) \tag{5}$$

for all positive integers k.

3. Borel's Theorem

We need a version of a classical theorem of Borel. Since it is generally stated in a slightly different form (the function f below is usually 0 or 1, and the conclusions are also slightly different from what we need) I include a complete proof. For Borel's original theorem, see [Bor97].

Theorem 1. *Let f and $u_1, \dots, u_n$ be entire functions of one variable satisfying*

$$\sum_i u_i = f. \tag{6}$$

If $u_1, \dots, u_n$ have no zeros then one of the following cases holds.

1. *$T(r, u_i) = O_{\mathrm{excl}}(T(r, f) + 1)$ for all i.*
2. *Some non-empty subsum $\sum_{i \in I} u_i = 0$.*

Proof. The proof is by induction n. If $n = 1$ then Case 1 holds automatically so there is nothing to prove. We assume that the theorem holds for sums with less than n terms.

If two terms in (6) are proportional, say $u_1 = au_2$, we can lump them together to get a shorter sum

$$(a+1)u_2 + u_3 + \cdots + u_n = f. \tag{7}$$

If $a = -1$ then case (2) holds. Otherwise we apply the theorem to this shorter sum and whether we get conclusion (1) or (2) for this sum we get the same conclusion for the original sum (6). We therefore assume that there are no proportional terms in the sum.

If we differentiate (6) we get

$$\sum_i \frac{u_i^{(j)}}{u_i} u_i = f^{(j)}. \quad \text{for } j = 0, \dots, n-1$$

This is a linear system for u_i. Two cases are possible.

1. $\det(u_i^{(j)}/u_i) \not\equiv 0$. Then we can solve for u_i and get

$$\begin{aligned} T(r, u_i) &= O(T(r, f)) + \sum_j O(T(r, u_i^{(j)}/u_i)) + O(1) \\ &= O(T(r, f)) + \sum_i o_{\mathrm{excl}}(T(r, u_i)) + O(1) \end{aligned}$$

   by LLD. We therefore have Case 1.
2. $\det(u_i^{(j)}/u_i) \equiv 0$. This means that the Wronskian $W(\{u_i\}) = \det(u_i^{(j)}) \equiv 0$. The theory of ordinary differential equations now says there are constants c_i such that

$$\sum_i c_i u_i = 0. \tag{8}$$

   We choose the shortest possible such sum, that is, the sum with the smallest number of non-zero c_i. Possibly after a reordering and a scaling we may assume that $c_1 = -1$, so that $u_1 = \sum_{i>1} c_i u_i$. Division by u_1 gives

$$\sum_{i>1} c_i u_i / u_1 = 1. \tag{9}$$

We set $v_i = c_i u_i / u_1$ and apply the theorem to the expression (9), which has less than n terms. There are two cases.

(a) $T(v_i, r) = O_{\text{excl}}(T(r,1)+1) = O_{\text{excl}}(1)$ for all i. Then all v_i are constant and $c_i u_i = a_i u_1$ for some a_i. We assumed that there were no proportional terms so this case is excluded.

(b) $\sum_{i \in I} v_i = 0$ for some set I. Then $\sum_{i \in I} c_i u_i = 0$ and this sum is shorter than (8). This is a contradiction. □

4. Vector fields

We start with the notation. Let p_i be Laurent polynomials and

$$\dot{z}_i = z_i p_i(z) \quad i = 1, \dots, n \tag{10}$$

be a vector field on $(\mathbb{C}^*)^n$. Write $p_i(Z) = \sum_\alpha p_{i,\alpha} Z^\alpha$ for each i. Let $\mathcal{M}$ be the multiplicative group generated by $\{Z^\alpha : p_{i,\alpha} \neq 0 \text{ for some } i\}$. $\mathcal{M}$ is isomorphic to a lattice in $\mathbb{Z}^n$ under the mapping $\mathbb{Z}^n \ni \alpha \mapsto Z^\alpha \in \mathcal{M}$. As such it has rank at most n. Let $\operatorname{rank} \mathcal{M} = m$ and $W_i = \prod Z_j^{a_{ij}}$ for $i = 1, \dots, m$ be a basis for $\mathcal{M}$. There are Laurent-polynomials f_i such that $p_i(Z) = f_i(W)$, where $W = (W_1, \dots, W_m)$. The following theorem tells us when the field is complete.

Theorem 2. *Notation as above, we have two cases.*

1. $m = n$. *Then (10) is not complete.*
2. $m < n$. *Then (10) is complete if and only if*

$$\dot{w}_i = w_i \sum_j a_{ij} f_j(w), \quad i = 1, \dots, m \tag{11}$$

is complete.

Proof. We first deal with the case $m = n$. We assume that (10) is complete and want to derive a contradiction. Choose c so that $\sum_{\alpha \in A} p_{i,\alpha} c^\alpha \neq 0$ for all non-zero subpolynomials and all indices i. Let $z(t)$ be the integral curve with $z(0) = c$. Apply Borel's theorem to $\sum_\alpha p_{i,\alpha} z^\alpha = \dot{z}_i / z_i$. Because of our choice of initial condition, Case 2 does not hold. We therefore have Case 1 and

$$T(r, z^\alpha) = O_{\text{excl}}(T(r, \dot{z}_i/z_i) + 1) = o_{\text{excl}}(T(r, z_i)) + O(1) = o_{\text{excl}}(T(r)) + O(1)$$

for all α such that $p_{i,\alpha} \neq 0$, where $T(r) = \max(T(r, z_i))$. It follows from the rules (3) that each element u in $\mathcal{M}$ satisfies $T(r, u) = o_{\text{excl}}(T(r)) + O(1)$. Since the rank of $\mathcal{M} = n$, for each i there is an integer d_i such that $Z^{d_i} \in \mathcal{M}$. Therefore, $T(r) = \max T(r, z_i) = o_{\text{excl}}(T(r)) + O(1)$. This implies $T(r, z_i) = O_{\text{excl}}(1)$ for all i, so z_i is constant for each i, and by (10), $p_i(z) = 0$ for all i. This is impossible since we chose $c = z(0)$ such that (in particular) $p_i(c) \neq 0$ for all i.

We now take the case $\operatorname{rank} \mathcal{M} < n$. Assume first that (11) is complete. Then w_i are entire functions and z_i satisfies

$$\dot{z}_i / z_i = p_i(z) = f_i(w)$$

The right hand sides are entire functions so integration gives

$$z_i(t) = e^{\int f_i(t) dt},$$

so (10) is complete.

If (10) is complete, set $w_i = \prod z_j^{a_{ij}}$. These are then entire function and they satisfy

$$\frac{\dot{w}_i}{w_i} = \sum_j a_{ij} \frac{\dot{z}_j}{z_j} = \sum_j a_{ij} p_j(z) = \sum_j a_{ij} f_j(w).$$

This is (11), which is therefore complete. □

Corollary 1. *All complete rational holomorphic vector fields on $(\mathbb{C}^*)^n$ preserve the volume form $\bigwedge dz_i/z_i$.*

Proof. The proof is by induction. We compute

$$\frac{d}{dt}\frac{dz_i}{z_i} = \frac{d\dot{z}_i\, z_i - \dot{z}_i\, dz_i}{z_i^2} = \frac{d(z_i p_i(z))z_i - z_i p_i(z) dz_i}{z_i^2} = d(p_i(z)).$$

This shows in particular that the result holds for $n = 1$. Also, we compute

$$\frac{d}{dt} \bigwedge_i \frac{dz_i}{z_i} = \sum_i \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{d}{dt}\frac{dz_i}{z_i} \wedge \cdots \wedge \frac{dz_n}{z_n} = \left(\sum_i z_i \frac{\partial p_i}{\partial z_i} \right) \left(\bigwedge_i \frac{dz_i}{z_i} \right).$$

Since the field is complete, we have Case 2 of the theorem. We use in particular that $p_i(z) = f_i(w)$. We have to prove that $\sum z_i \partial p_i/\partial z_i(z) = 0$, so we compute

$$\begin{aligned} \sum_i z_i \frac{\partial f_i}{\partial z_i} &= \sum_i z_i \sum_j \frac{\partial f_i}{\partial w_j}\frac{\partial w_j}{\partial z_i} = \sum_i z_i \sum_j \frac{\partial f_i}{\partial w_j} w_j \frac{a_{ji}}{z_i} \\ &= \sum_{i,j} \frac{\partial f_i}{\partial w_j} w_j a_{ji} = \sum_j \frac{\partial (af)_j}{\partial w_j} w_j, \end{aligned} \tag{12}$$

where $(af)_j = \sum a_{ji} f_i$. By Theorem 2, (11) is complete and by the induction hypothesis, the last expression is 0. The corollary is proved. □

Corollary 2. *All complete rational holomorphic vector fields on $(\mathbb{C}^*)^2$ are of form*

$$\begin{aligned} \dot{z}_1 &= z_1(a_2 f(z_1^{a_1} z_2^{a_2}) + c_1) \\ \dot{z}_2 &= -z_2(a_1 f(z_1^{a_1} z_2^{a_2}) + c_2), \end{aligned} \tag{13}$$

where f is a Laurent polynomial, a_1, a_2 are integers and c_1, c_2 are complex numbers. Conversely, all such vector fields are complete.

Proof. We use the notation in the theorem. If the field is complete, we must have $\dim \mathcal{M} = 1$, so $p_i(Z) = f_i(W)$ for some Laurent polynomial f and some monomial $W = Z_1^{a_1} Z_2^{a_2}$. The field (11) becomes

$$\dot{w} = w(a_1 f_1(w) + a_2 f_2(w)),$$

which is complete if and only if $a_1 f_1(w) + a_2 f_2(w)$ is constant. If we write $f_1 = a_2 f + c_1$ and $f_2 = -a_1 f + c_2$ we get the corollary. □

Remark . In the same way, we can derive (rather complicated) formulas for the complete vector fields on $(\mathbb{C}^*)^n$ for any n.

Remark . The analogs of Theorem 2 and Corollary 2 for non-rational fields are false. To get a counterexample, observe that

$$\begin{aligned} \dot z_1 &= 0 \\ \dot z_2 &= -z_2 z_1 \end{aligned} \tag{14}$$

is complete by Corollary 2. Also, the mapping

$$\begin{aligned} z_1 &= \zeta_1 e^{\zeta_1 \zeta_2} \\ z_2 &= \zeta_2 e^{-\zeta_1 \zeta_2} \end{aligned}$$

is a bijection of $(\mathbb{C}^*)^2$ (it is the time 1 flow of the field $\dot\zeta_1 = \zeta_1^2 \zeta_2$, $\dot\zeta_2 = -\zeta_1 \zeta_2^2$). If we express (14) in the new coordinates we get

$$\begin{aligned} \dot\zeta_1 &= \zeta_1^3 \zeta_2 e^{\zeta_1 \zeta_2} \\ \dot\zeta_2 &= -\zeta_1 \zeta_2 (1 + \zeta_1 \zeta_2) e^{\zeta_1 \zeta_2}, \end{aligned}$$

which is not of a form corresponding to (13).

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